



TITLE:

Jaynes-Cummings modelのexactly solvableな拡張について (幾何学的力学系の新展開)

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CITATION:

Suzuki, Tatsuo. Jaynes-Cummings modelのexactly solvableな拡張について (幾何学的力学系の新展開). 数理解析研究所講究録 2012, 1774: 163-171

ISSUE DATE:

2012-01

URL:

<http://hdl.handle.net/2433/171723>

RIGHT:

Jaynes-Cummings model の exactly solvable な拡張について

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Abstract

As an application of noncommutative characteristic equations for matrices with noncommutative entries, we study the exact-solvability of Jaynes-Cummings model (JCM) and its extended models in the view of “operator-valued eigenvalues”. Then we find a new method for obtaining invariant subspaces of given models and obtain the evolution operator of an extended JCM Hamiltonian by using the NC spectral decomposition method. Next, we define a class of exactly-solvable matrix Hamiltonian and obtain an infinite number of finite-dimensional invariant subspaces.

1 Introduction

The original Jaynes-Cummings model (JCM) is defined by the Hamiltonian [1]

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a \\ \rho a^\dagger & \hbar\omega a^\dagger a - \frac{\epsilon}{2} \end{pmatrix} \quad (1)$$

where ρ is a real parameter. Note here that the Hamiltonian H is hermitian.

In previous paper [1], they considered an extension of JCM Hamiltonian in the form

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2} & \rho a^k \\ \phi \rho (a^\dagger)^k & \hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2} \end{pmatrix} \quad (2)$$

where $k \in \mathbf{N}$, $\phi = \pm 1$ and $P(a^\dagger a)$ denotes a polynomial of degree $d \geq 2$. If $\phi = 1$, H is hermitian and $\phi = -1$, H is nonhermitian.

In [1], they found H preserved an infinite number of finite-dimensional subspaces

$$\mathcal{V}_n = \text{span} \left\{ \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \quad (3)$$

Therefore H is exactly solvable.

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Since

$$H \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} (\hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2})|n\rangle \\ \phi\rho(a^\dagger)^k|n\rangle \end{pmatrix} = \begin{pmatrix} (\hbar\omega n + P(n) + \frac{\epsilon}{2})|n\rangle \\ \phi\rho\sqrt{(n+1)(n+2)\cdots(n+k)}|n+k\rangle \end{pmatrix},$$

$$H \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} = \begin{pmatrix} \rho a^k|n+k\rangle \\ (\hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2})|n+k\rangle \end{pmatrix} = \begin{pmatrix} \rho\sqrt{(n+k)(n+k-1)\cdots(n+1)}|n\rangle \\ (\hbar\omega(n+k) + P(n+k) - \frac{\epsilon}{2})|n+k\rangle \end{pmatrix},$$

the Hamiltonian matrix is

$$H_{n+k} := \begin{pmatrix} \hbar\omega n + P(n) + \frac{\epsilon}{2} & \rho\sqrt{(n+k)(n+k-1)\cdots(n+1)} \\ \phi\rho\sqrt{(n+1)(n+2)\cdots(n+k)} & \hbar\omega(n+k) + P(n+k) - \frac{\epsilon}{2} \end{pmatrix}. \quad (4)$$

For simplicity, they imposed $P(n) = P(n+k) = 0$. Then the eigenvalues of H_{n+k} are

$$\lambda_{n+k}^I = \frac{\hbar\omega(2n+k) + \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2(n+1)\cdots(n+k)}}{2}, \quad (5)$$

$$\lambda_{n+k}^{II} = \frac{\hbar\omega(2n+k) - \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2(n+1)\cdots(n+k)}}{2} \quad (6)$$

(equation (34) in [1]).

2 Review of NC Spectral Decomposition

2.1 NC Version of the Characteristic Polynomial

First, we review the noncommutative (NC) version of the characteristic polynomial.

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, we put $A^n = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix}$ ($n = 0, 1, 2$). Then, we denote $\Phi_1(\lambda), \Phi_2(\lambda)$ as two polynomials given by

$$\begin{aligned} \Phi_i(\lambda) &= \begin{vmatrix} a_{i1}^{(2)} & a_{i2}^{(2)} & \boxed{\lambda^2} \\ a_{i1}^{(1)} & a_{i2}^{(1)} & \lambda \\ a_{i1}^{(0)} & a_{i2}^{(0)} & 1 \end{vmatrix} \\ &= \lambda^2 - (a_{i1}^{(2)}, a_{i2}^{(2)}) \begin{pmatrix} a_{i1}^{(1)} & a_{i2}^{(1)} \\ a_{i1}^{(0)} & a_{i2}^{(0)} \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ 1 \end{pmatrix}. \end{aligned}$$

More explicitly,

$$\begin{aligned} \Phi_1(\lambda) &= \lambda^2 - (a_{11} + a_{12}a_{22}a_{12}^{-1})\lambda + (a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21}), \\ \Phi_2(\lambda) &= \lambda^2 - (a_{22} + a_{21}a_{11}a_{21}^{-1})\lambda + (a_{21}a_{11}a_{21}^{-1}a_{22} - a_{21}a_{12}). \end{aligned}$$

We remark that the NC Cayley-Hamilton's theorem holds ;

$$A^2 - \begin{pmatrix} a_{11} + a_{12}a_{22}a_{12}^{-1} & 0 \\ 0 & a_{22} + a_{21}a_{11}a_{21}^{-1} \end{pmatrix} A + \begin{pmatrix} a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21} & 0 \\ 0 & a_{21}a_{11}a_{21}^{-1}a_{22} - a_{21}a_{12} \end{pmatrix} = O.$$

Proposition 1 (A sufficient condition for no-existence of inverse elements).
 For $i = 1, 2$, let $F_i(x)$ be entire functions of x . If relations

$$a_{12}a_{22} = F_1(a_{22})a_{12}, \quad a_{21}a_{11} = F_2(a_{11})a_{21} \quad (7)$$

hold, the noncommutative characteristic polynomials $\Phi_1(\lambda), \Phi_2(\lambda)$ are given by

$$\begin{aligned} \Phi_1(\lambda) &= \lambda^2 - (a_{11} + F_1(a_{22}))\lambda + (F_1(a_{22})a_{11} - a_{12}a_{21}), \\ \Phi_2(\lambda) &= \lambda^2 - (a_{22} + F_2(a_{11}))\lambda + (F_2(a_{11})a_{22} - a_{21}a_{12}). \end{aligned}$$

For example, $a_{12} = a$, $a_{22} = N = a^\dagger a$ and by using the relation $[a, a^\dagger] = 1$, then

$$a_{12}a_{22} = aN = aa^\dagger a = (N + 1)a = (a_{22} + 1)a_{12}.$$

2.2 NC Spectral Decomposition

Let α_1, α_2 be two solutions of the NC characteristic equation $\Phi_1(\lambda) = 0$ and β_1, β_2 two solutions of $\Phi_2(\lambda) = 0$ respectively. We put

$$x_1 = \begin{pmatrix} \alpha_1 & \\ & \beta_1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} \alpha_2 & \\ & \beta_2 \end{pmatrix} \quad (8)$$

and

$$P_1 = (x_1 - x_2)^{-1}(A - x_2), \quad (9)$$

$$P_2 = (x_2 - x_1)^{-1}(A - x_1), \quad (10)$$

then we have the NC spectral decomposition [3]

$$A^n = x_1^n P_1 + x_2^n P_2 \quad (n = 0, 1, 2, \dots). \quad (11)$$

3 Operator Method for the Eigenvalue Problem of the Exactly Solvable Hamiltonian (2)

3.1 Operator-valued Eigenvalues

For the eigenvalue problem of the extended JCM Hamiltonian (2)

$$H = \begin{pmatrix} \hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2} & \rho a^k \\ \phi \rho (a^\dagger)^k & \hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2} \end{pmatrix},$$

we study another approach. First, we consider a Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix} \quad (12)$$

where $N = a^\dagger a$, $c_{ij} \in \mathbf{R}$ and $f_i(N)$ ($i = 1, 2$) are some functions of N . Then we have a proposition as follows;

Proposition 2. *NC characteristic equations of*

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix} \quad (13)$$

are

$$\Phi_1(\lambda) = \lambda^2 - (f_1(N) + f_2(N+k))\lambda + f_1(N)f_2(N+k) - c_{12}c_{21}a^k(a^\dagger)^k = 0, \quad (14)$$

$$\Phi_2(\lambda) = \lambda^2 - (f_1(N-k) + f_2(N))\lambda + f_1(N-k)f_2(N) - c_{12}c_{21}(a^\dagger)^ka^k = 0. \quad (15)$$

The operator solutions α_+, α_- of (14) and β_+, β_- of (15) are

$$\alpha_\pm = \frac{1}{2} \left\{ f_1(N) + f_2(N+k) \pm \sqrt{(f_1(N) - f_2(N+k))^2 + 4c_{12}c_{21}a^k(a^\dagger)^k} \right\}, \quad (16)$$

$$\beta_\pm = \frac{1}{2} \left\{ f_1(N-k) + f_2(N) \pm \sqrt{(f_1(N-k) - f_2(N))^2 + 4c_{12}c_{21}(a^\dagger)^ka^k} \right\}. \quad (17)$$

Remark 1. These α_+, α_- and β_+, β_- are considered as “operator-valued eigenvalues”.

3.2 A New Method for Obtaining Invariant Subspaces

Since $a^k(a^\dagger)^k = (N+1)(N+2)\cdots(N+k)$ and $(a^\dagger)^k a^k = N(N-1)\cdots(N-k+1)$, we note that $\alpha_\pm = \alpha_\pm(N)$, $\beta_\pm = \beta_\pm(N)$ and

$$\beta_\pm(N+k) = \alpha_\pm(N). \quad (18)$$

If we put $f_1(N) = \hbar\omega N + P(N) + \frac{\epsilon}{2}$, $f_2(N) = \hbar\omega N + P(N) - \frac{\epsilon}{2}$, $c_{12} = \rho$, $c_{21} = \phi\rho$, we have

$$\alpha_\pm(N) = \frac{1}{2} \left\{ \hbar\omega(2N+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 a^k (a^\dagger)^k} \right\} \quad (19)$$

(here we omitted $P(N)$ and $P(N+k)$ for simplicity). Therefore

$$\alpha_\pm(N)|n\rangle = \frac{1}{2} \left\{ \hbar\omega(2N+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 a^k (a^\dagger)^k} \right\} |n\rangle \quad (20)$$

$$= \frac{1}{2} \left\{ \hbar\omega(2n+k) \pm \sqrt{(\hbar\omega k - \epsilon)^2 + 4\phi\rho^2 (n+1)\cdots(n+k)} \right\} |n\rangle. \quad (21)$$

These values are nothing but λ_{n+k}^I , λ_{n+k}^{II} . By using $\beta_\pm(N+k) = \alpha_\pm(N)$, we have

$$\beta_\pm(N)|n+k\rangle = \beta_\pm(n+k)|n+k\rangle = \alpha_\pm(n)|n+k\rangle = \lambda_{n+k}^{I, II}|n+k\rangle \quad (22)$$

Since α_\pm is derived from $\Phi_1(\lambda) = 0$ (1st row) and β_\pm is derived from $\Phi_2(\lambda) = 0$ (2nd row), we can find a basis

$$\left\{ \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} \right\}$$

of an invariant subspace and eigenvalues λ_{n+k}^I , λ_{n+k}^{II} .

3.3 Evolution Operator and NC Spectral Decomposition Method

Next, we calculate the time evolution operator of H . By the NC spectral decomposition method, the projection operators of H are

$$P_1 = \begin{pmatrix} (\alpha_+ - \alpha_-)^{-1} & \\ & (\beta_+ - \beta_-)^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \alpha_- & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \beta_- \end{pmatrix}, \quad (23)$$

$$P_2 = \begin{pmatrix} (\alpha_- - \alpha_+)^{-1} & \\ & (\beta_- - \beta_+)^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \alpha_+ & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \beta_+ \end{pmatrix}. \quad (24)$$

Here we put

$$\beta_+(N) - \beta_-(N) = \sqrt{(f_1(N-k) - f_2(N))^2 + 4c_{12}c_{21}(a^\dagger)^k a^k} =: \sqrt{D(N)} \quad (25)$$

and

$$\lambda_\pm(N) := \beta_\pm(N), \quad (26)$$

then

$$x_1 = \begin{pmatrix} \alpha_+(N) & \\ & \beta_+(N) \end{pmatrix} = \begin{pmatrix} \lambda_+(N+k) & \\ & \lambda_+(N) \end{pmatrix}, \quad x_2 = \begin{pmatrix} \lambda_-(N+k) & \\ & \lambda_-(N) \end{pmatrix}, \quad (27)$$

$$P_1 = \begin{pmatrix} (\sqrt{D(N+k)})^{-1} & \\ & (\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_-(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_-(N) \end{pmatrix}, \quad (28)$$

$$P_2 = \begin{pmatrix} (-\sqrt{D(N+k)})^{-1} & \\ & (-\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_+(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_+(N) \end{pmatrix}. \quad (29)$$

Therefore $H = x_1 P_1 + x_2 P_2$ implies

$$\begin{aligned} e^{itH} &= e^{itx_1} P_1 + e^{itx_2} P_2 \\ &= \begin{pmatrix} e^{it\lambda_+(N+k)} & \\ & e^{it\lambda_+(N)} \end{pmatrix} \begin{pmatrix} (\sqrt{D(N+k)})^{-1} & \\ & (\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_-(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_-(N) \end{pmatrix} \\ &\quad + \begin{pmatrix} e^{it\lambda_-(N+k)} & \\ & e^{it\lambda_-(N)} \end{pmatrix} \begin{pmatrix} (-\sqrt{D(N+k)})^{-1} & \\ & (-\sqrt{D(N)})^{-1} \end{pmatrix} \begin{pmatrix} f_1(N) - \lambda_+(N+k) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) - \lambda_+(N) \end{pmatrix} \\ &= \begin{pmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{pmatrix}, \end{aligned} \quad (30)$$

where

$$(1,1) = e^{\frac{it}{2}\{f_1(N)+f_2(N+k)\}} \left\{ \cos \frac{t}{2} \sqrt{D(N+k)} + i(f_1(N) - f_2(N+k)) \frac{\sin \frac{t}{2} \sqrt{D(N+k)}}{\sqrt{D(N+k)}} \right\} \quad (31)$$

$$(2,2) = e^{\frac{it}{2}\{f_1(N-k)+f_2(N)\}} \left\{ \cos \frac{t}{2} \sqrt{D(N)} - i(f_1(N-k) - f_2(N)) \frac{\sin \frac{t}{2} \sqrt{D(N)}}{\sqrt{D(N)}} \right\} \quad (32)$$

$$(1,2) = 2i c_{12} e^{\frac{it}{2}\{f_1(N)+f_2(N+k)\}} \frac{\sin \frac{t}{2} \sqrt{D(N+k)}}{\sqrt{D(N+k)}} a^k \quad (33)$$

$$(2,1) = 2i c_{21} e^{\frac{it}{2}\{f_1(N-k)+f_2(N)\}} \frac{\sin \frac{t}{2} \sqrt{D(N)}}{\sqrt{D(N)}} (a^\dagger)^k. \quad (34)$$

For example, we consider the JCM Hamiltonian (from Wikipedia notation)

$$H_{JC} = \hbar \begin{pmatrix} \nu a^\dagger a + \frac{\omega}{2} & \frac{\Omega}{2} a \\ \frac{\Omega}{2} a^\dagger & \nu a^\dagger a - \frac{\omega}{2} \end{pmatrix}. \quad (35)$$

If we put

$$f_1(N) = \nu N + \frac{\omega}{2}, \quad f_2(N) = \nu N - \frac{\omega}{2}, \quad c_{12} = c_{21} = \frac{\Omega}{2} =: g, \quad k = 1, \quad t \mapsto -t$$

in (30), since

$$f_1(N) + f_2(N+1) = 2\nu \left(N + \frac{1}{2}\right), \quad f_1(N) - f_2(N+1) = \omega - \nu =: \delta, \quad (36)$$

$$D(N+1) = \delta^2 + 4g^2(N+1) = 4 \left(\frac{\delta^2}{4} + g^2 N + g^2 \right) =: 4(\varphi + g^2), \quad (37)$$

we obtain the famous result

$$= e^{-itH_{JC}/\hbar} \begin{pmatrix} e^{-i\nu t(N+\frac{1}{2})} \left(\cos t \sqrt{\varphi + g^2} - \frac{i\delta}{2} \frac{\sin t \sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} \right) & -ig e^{-i\nu t(N+\frac{1}{2})} \frac{\sin t \sqrt{\varphi + g^2}}{\sqrt{\varphi + g^2}} a \\ -ig e^{-i\nu t(N-\frac{1}{2})} \frac{\sin t \sqrt{\varphi}}{\sqrt{\varphi}} a^\dagger & e^{-i\nu t(N-\frac{1}{2})} \left(\cos t \sqrt{\varphi} + \frac{i\delta}{2} \frac{\sin t \sqrt{\varphi}}{\sqrt{\varphi}} \right) \end{pmatrix}. \quad (38)$$

4 A Generalization of JCM Hamiltonian to $M \times M$ Matrix

We can observe that the extended JCM Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^k \\ c_{21}(a^\dagger)^k & f_2(N) \end{pmatrix}$$

has an infinite number of finite-dimensional invariant subspaces

$$\mathcal{V}_n = \text{span} \left\{ \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}.$$

If we consider a 3×3 matrix Hamiltonian

$$H = \begin{pmatrix} f_1(N) & c_{12}a^{k_1} & c_{13}a^{k_1+k_2} \\ c_{21}(a^\dagger)^{k_1} & f_2(N) & c_{23}a^{k_2} \\ c_{31}(a^\dagger)^{k_1+k_2} & c_{32}(a^\dagger)^{k_2} & f_3(N) \end{pmatrix}, \quad (39)$$

we can find an infinite number of finite-dimensional invariant subspaces

$$\mathcal{V}_{n;k_1,k_2} = \text{span} \left\{ \begin{pmatrix} |n\rangle \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k_1\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ |n+k_1+k_2\rangle \end{pmatrix} \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \quad (40)$$

We can generalize these matrix Hamiltonians in this way;

Theorem 3. *A $M \times M$ matrix Hamiltonian*

$$H = \begin{pmatrix} f_1(a^\dagger a) & & c_{1j}a^{\sum_{l=1}^{j-1} k_l} \\ & \ddots & \\ c_{ij}(a^\dagger)^{\sum_{l=j}^{i-1} k_l} & & f_M(a^\dagger a) \end{pmatrix} \quad (41)$$

preserves an infinite number of finite-dimensional subspaces

$$\mathcal{V}_{n;k_1,\dots,k_{M-1}} := \text{span}_{1 \leq j \leq M} \left\{ |n + \sum_{l=1}^{j-1} k_l\rangle \otimes \mathbf{e}_j \right\}, \quad \forall n \in \mathbf{N} \cup \{0\}. \quad (42)$$

Therefore, H is exactly solvable.

The corresponding Hamiltonian matrix restricted to $\mathcal{V}_{n;k_1,\dots,k_{M-1}}$ is

$$H_{n;k_1,\dots,k_{M-1}} := \begin{pmatrix} f_1(n) & & c_{1j} \sqrt{(n+\sum_{l=1}^{j-1} k_l) \cdots (n+\sum_{l=1}^{j-1} k_l+1)} \\ & \ddots & \\ c_{ij} \sqrt{(n+\sum_{l=1}^{j-1} k_l+1) \cdots (n+\sum_{l=1}^{j-1} k_l)} & & f_M(n+\sum_{l=1}^{M-1} k_l) \end{pmatrix} \quad (43)$$

Proof. By using

$$a^k |n + k' \rangle = \sqrt{(n + k') \cdots (n + k' - k + 1)} |n + k' - k \rangle, \quad (44)$$

$$(a^\dagger)^k |n + k' \rangle = \sqrt{(n + k' + 1) \cdots (n + k' + k)} |n + k' + k \rangle, \quad (45)$$

we have

$$\begin{aligned} & H |n + \sum_{l=1}^{j-1} k_l \rangle \otimes \mathbf{e}_j \\ &= \begin{pmatrix} c_{ij} a^{\sum_{l=i}^{j-1} k_l} |n + \sum_{l=1}^{j-1} k_l \rangle & (\text{if } i < j) \\ f_i(a^\dagger a) |n + \sum_{l=1}^{j-1} k_l \rangle & (\text{if } i = j) \\ c_{ij} (a^\dagger)^{\sum_{l=j}^{i-1} k_l} |n + \sum_{l=1}^{j-1} k_l \rangle & (\text{if } i > j) \end{pmatrix} \\ &= \begin{pmatrix} c_{ij} \sqrt{(n + \sum_{l=1}^{j-1} k_l) \cdots (n + \sum_{l=1}^{i-1} k_l + 1)} |n + \sum_{l=1}^{i-1} k_l \rangle \\ f_i(n + \sum_{l=1}^{i-1} k_l) |n + \sum_{l=1}^{i-1} k_l \rangle \\ c_{ij} \sqrt{(n + \sum_{l=1}^{j-1} k_l + 1) \cdots (n + \sum_{l=1}^{i-1} k_l)} |n + \sum_{l=1}^{i-1} k_l \rangle \end{pmatrix} \end{aligned} \quad (46)$$

□

5 Discussion

We studied the exact-solvability of Jaynes-Cummings model and its extened models in the view of “operator-valued eigenvalues”. Analysing relations among “eigenvalues of each row”, we found a new method for obtaining invariant subspaces of given models. Then we obtained the evolution operator of an extended JCM Hamiltonian by using the NC spectral decomposition method. Next, we defined a class of exactly-solvable matrix Hamiltonian and obtained an infinite number of finite-dimensional invariant subspaces.

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